Modelling fractured porous media

Towards a discrete fracture model using a cell-centered MPFA finite volume scheme

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Exemplary applications

picture: ucsusa.org
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Exemplary applications
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Modelling of flow through fractured rock

picture: hinderedsettling.com
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Modelling of flow through fractured rock

Motivation

Fracture representation in dfm models
Motivation

*Control-volume formulation of diffusion equations*

Diffusion problem:

\[-\nabla \cdot (K \nabla u) = f, \text{ in } \Omega\]

\[u = \bar{u}, \text{ on } \Gamma_D\]

\[K \nabla u \cdot n = g, \text{ on } \Gamma_N\]

Finite volume formulation:

\[\int_{\partial V} -K(x) \nabla u(x) \cdot n_v(x) \, dS = \sum_\sigma \bar{F}_{V,\sigma} = \int_V f(x) \, dx\]
Motivation

Flux approximation

Two-point flux approximation:

\[ f_\sigma \approx T_\sigma (u_1 - u_2) \]

Multi-point flux approximation:

\[ f_\sigma \approx \sum_{j \in J} T_{\sigma j} u_j \]
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FVCA5 - test 3: oblique flow

FV TPFA method

Reference solution [2]
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Flux calculation
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*The dual grid*
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Interaction regions
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*Calculation of transmissivity coefficients*

**Conditions on sub faces:** [1]
- Continuity of flux
- Continuity of potential

**Conditions on sub volumes:** [1]
- Potential has to match cell center value
Mpfa-O method

*Expression of sub volume face fluxes*

Flux over face i in cell j:

\[
f_i = -n_i^T K_j \nabla U_j
\]

Linear potential in cell j:

\[
U_j(x) = \nabla U_j \cdot (x - x_{j0}) + u_{j0}
\]

\[
\rightarrow \tilde{u}_{jk} = \nabla U_j \cdot (x_{jk} - x_{j0}) + u_{j0}
\]
Mpfa-O method

Sub volume gradient

\[ \bar{u}_{jk} = \nabla U_j \cdot (x_{jk} - x_j0) + u_{j0} \]

\[ \begin{align*}
X_j \nabla U_j &= \begin{bmatrix} \bar{u}_{j1} - u_{j0} \\ \bar{u}_{j2} - u_{j0} \end{bmatrix}, \\
X_j &= \begin{bmatrix} (x_{j1} - x_{j0})^T \\ (x_{j2} - x_{j0})^T \end{bmatrix}
\]
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*Sub volume gradient*

Introducing...

\[
R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{v}_j^1 = R(x_{j2} - x_{j0}), \quad \mathbf{v}_j^2 = -R(x_{j1} - x_{j0}),
\]

we can write:

\[
X_j^{-1} = \frac{1}{T_j} [\mathbf{v}_j^1, \mathbf{v}_j^2], \quad T_j = (x_{j1} - x_{j0})^T R(x_{j2} - x_{j0})
\]
Mpfa-O method

*Sub volume gradient*

Expression for the gradient of $u$ in scv $j$:

$$
\nabla U_j = \frac{1}{T_j} \sum_{k=1}^{2} v_{jk} (\bar{u}_{jk} - u_{j0}),
$$

and for the fluxes over sub face $i$ in scv $j$:

$$
f_i = \sum_{k=1}^{2} - \frac{n_i^T K_j v_{jk}}{T_j} (\bar{u}_{jk} - u_{j0}) = \sum_{k=1}^{2} \omega_{ijk} (\bar{u}_{jk} - u_{j0})$$
Mpfa-O method
Transmissivity coefficients

Final system of equations:

\[ f_1 = \omega_{111}(\bar{u}_1 - u_1) + \omega_{112}(\bar{u}_4 - u_1) = \omega_{121}(\bar{u}_2 - u_2) + \omega_{122}(\bar{u}_1 - u_2), \]
\[ f_2 = \omega_{221}(\bar{u}_2 - u_2) + \omega_{222}(\bar{u}_1 - u_2) = \omega_{231}(\bar{u}_3 - u_3) + \omega_{232}(\bar{u}_2 - u_3), \]
\[ f_3 = \omega_{331}(\bar{u}_3 - u_3) + \omega_{332}(\bar{u}_2 - u_3) = \omega_{341}(\bar{u}_1 - u_4) + \omega_{342}(\bar{u}_3 - u_4), \]
\[ f_4 = \omega_{441}(\bar{u}_4 - u_4) + \omega_{442}(\bar{u}_3 - u_4) = \omega_{411}(\bar{u}_1 - u_1) + \omega_{412}(\bar{u}_4 - u_1). \]
Mpfa-O method

Transmissivity coefficients

\[ f_1 = \omega_{111}(\bar{u}_1 - u_1) + \omega_{112}(\bar{u}_4 - u_1) = \omega_{121}(\bar{u}_2 - u_2) + \omega_{122}(\bar{u}_1 - u_2), \]
\[ f_2 = \omega_{221}(\bar{u}_2 - u_2) + \omega_{222}(\bar{u}_1 - u_2) = \omega_{231}(\bar{u}_3 - u_3) + \omega_{232}(\bar{u}_2 - u_3), \]
\[ f_3 = \omega_{331}(\bar{u}_3 - u_3) + \omega_{332}(\bar{u}_2 - u_3) = \omega_{341}(\bar{u}_1 - u_4) + \omega_{342}(\bar{u}_3 - u_4), \]
\[ f_4 = \omega_{441}(\bar{u}_4 - u_4) + \omega_{442}(\bar{u}_3 - u_4) = \omega_{411}(\bar{u}_1 - u_1) + \omega_{412}(\bar{u}_4 - u_1). \]

\[ f = C v - D u \]
\[ A v = B u \]
\[ \rightarrow f = (C A^{-1} B - D) u = T u \]
Mpfa-O method
Formulation for general grids

Boundary conditions

Random interaction volume setup
Mpfa-O method
Formulation for general grids

\[ f_i = \omega_{ij+1}(\bar{u}_{j+1} - u_j) + \omega_{ij+2}(\bar{u}_{j+2} - u_j) \]
\[ = \omega_{ij-1}(\bar{u}_{j-1} - u_{j-1}) + \omega_{ij-2}(\bar{u}_{j-2} - u_{j-1}), \]
\[ f_N = \omega_{iNj+1}(\bar{u}_N - u_j) + \omega_{iNj+2}(\bar{u}_{j+2} - u_j) = \bar{f}_N, \]
\[ f_D = \omega_{iDj+1}(\bar{u}_D - u_j) + \omega_{iDj+2}(\bar{u}_{j+2} - u_j). \]

\[ f(|i+N+D| \times 1) = C(|i+N+D| \times |i+N|) v(|i+N| \times 1) - D(|i+N+D| \times |j+D|) u(|j+D| \times 1) \]
\[ A(|i+N| \times |i+N|) v(|i+N| \times 1) = B(|i+N| \times |j+D|) u(|j+D| \times 1) + f(|i+N| \times 1) \]
\[ \rightarrow f = \left( C A^{-1} B - D \right) u + C A^{-1} f \]
Mpfa-O method

FVCA5 - test 3: oblique flow

MPFA-O method

Reference solution [2]
Mpfa-O method

Test for two phase flow

Grid and boundary conditions

\[ p_1 > p_2 \]
\[ S_n = 1 \]

\[ p_2 \]
\[ S_n = 0 \]

FV TPFA method

MPFA-O method
Mpfa-O method
*Treatment of interior boundaries*

- interior Dirichlet face
- interior Flux face
Mpfa–O method

Treatment of interior boundaries

\[ f_i = \omega_{ij+1}(\bar{u}_{j+1} \rightarrow i - u_j) + \omega_{ij+2}(\bar{u}_{j+2} \rightarrow i - u_j) \]
\[ = \omega_{ij-1}(\bar{u}_{j-1} \rightarrow i - u_j) + \omega_{ij-2}(\bar{u}_{j-2} \rightarrow i - u_j), \]
\[ f_{N+} = \omega_{iN+j+1}(\bar{u}_{N+} - u_j) + \omega_{iN+j+2}(\bar{u}_{j+2} \rightarrow i - u_j) = \tilde{f}_{N+}, \]
\[ f_{N-} = \omega_{iN-j+1}(\bar{u}_{N-} - u_j) + \omega_{iN-j+2}(\bar{u}_{j+2} \rightarrow i - u_j) = \tilde{f}_{N-}, \]
\[ f_{D+} = \omega_{iD+j+1}(\bar{u}_{D+} - u_j) + \omega_{iD+j+2}(\bar{u}_{j+2} \rightarrow i - u_j), \]
\[ f_{D-} = \omega_{iD-j+1}(\bar{u}_{D-} - u_j) + \omega_{iD-j+2}(\bar{u}_{j+2} \rightarrow i - u_j). \]

\[
\begin{align*}
\mathbf{f}(|i+N+D| \times 1) &= \mathbf{C}(|i+N+D| \times |i+N|) \mathbf{v}(|i+N| \times 1) - \mathbf{D}(|i+N+D| \times |j+D|) \mathbf{u}(|j+D| \times 1) \\
\mathbf{A}(|i+N| \times |i+N|) \mathbf{v}(|i+N| \times 1) &= \mathbf{B}(|i+N| \times |j+D|) \mathbf{u}(|j+D| \times 1) + \mathbf{f}(|i+N| \times 1) \\
\rightarrow \quad \mathbf{f} &= \left(C A^{-1} B - D\right) \mathbf{u} + CA^{-1} \tilde{\mathbf{f}}
\end{align*}
\]
Mpfa-O method
Interior Dirichlet boundaries - exemplary results

Problem Setup:

\[ p_w = 2\text{bar} \quad S_n = 0 \]

\[ p_w = 3\text{bar} \quad S_n = 1 \]

\[ p_w = 1\text{bar} \quad S_n = 0 \]
Mpfa-O method

*Interior Dirichlet boundaries - exemplary results*

Results - saturation:
Mpfa-O method

*Interior Flux boundaries – exemplary results*

**Problem Setup:**

- $p_w = 2\text{bar}$
- $S_n = 1$
- $f_{N+} = 0$
- $f_{N-} = 0$
- Internal barrier
- $p_w = 1\text{bar}$
- $S_n = 0$

Here, the diagram illustrates the setup with interior flux boundaries and exemplary results.
Mpfa-O method

*Interior Flux boundaries - exemplary results*

Results - saturation:
Mpfa-O method

*Interior Flux boundaries – exemplary results*

Results – wetting phase pressure:
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Multi-point flux approximation methods

Mpfa – discrete fracture model

Summary and Outlook
Mpfa - dfm

*Lower dimensional fracture elements*
Mpfα - dfm

Model problem

\[
\frac{\partial (\rho \phi_m)}{\partial t} - \nabla \cdot (\rho K_m \nabla u) = 0, \quad \text{in } \Omega_i, \quad i = 1, 2
\]

\[
\frac{\partial (\bar{\rho} \alpha \phi)}{\partial t} - \nabla_{\tau} \cdot (a \rho K_{f,\tau} \nabla_{\tau} \bar{U}_f) = -\rho \left( v_{n,\gamma_2} n_{\gamma_2} + v_{n,\gamma_1} n_{\gamma_1} \right), \quad \text{in } \gamma
\]

\[
-\bar{\xi} v_{n,\gamma_1} n_{\gamma_1} + \alpha_f \bar{u}_{\gamma_1} = -(1 - \bar{\xi}) v_{n,\gamma_2} n_{\gamma_2} + \alpha_f \bar{U}_f, \quad \text{on } \gamma_1
\]

\[
-\bar{\xi} v_{n,\gamma_2} n_{\gamma_2} + \alpha_f \bar{u}_{\gamma_2} = -(1 - \bar{\xi}) v_{n,\gamma_1} n_{\gamma_1} + \alpha_f \bar{U}_f, \quad \text{on } \gamma_2
\]

\[
u_i = \bar{u}_i,
\]

\[
\bar{U}_f = \bar{\bar{u}}_f,
\]

\[
\text{on } \partial \gamma
\]
Mpfa – dfm
Approach 1: No pressure difference across fracture

Assumption $\Delta p \approx 0$ justified when:

- $K_{f,\eta} \gg K_m$
- $a \approx \epsilon$

Iterative solution algorithm:

- while $||\Delta||_\infty > \epsilon$
  - obtain $u_{m}^{n+1}$ with $u_{f}^{m}$ as interior Dirichlet boundary condition
  - update $u_{i}^{m} \rightarrow u_{i}^{m+1}$ with matrix in-/outfluxes as sources in fracture
  - calculate $||\Delta||_\infty = \frac{||\Delta u_{m}||_\infty}{||u_{m}||_\infty} + \frac{||\Delta u_{f}||_\infty}{||u_{f}||_\infty}$
Mpfa - dfm

Test case 1

\[ \begin{align*}
K_m &= 0.01 K_f \\
p_m &= 2 \\
p_f &= 2 \\
K_f_1 &| K_f_2 | K_f_1 \\
p_f &= 1 \\
p_m &= 1
\end{align*} \]
Mpfa - dfm

Test case 1.1: homogeneous fracture permeability
Mpfa - dfm

Test case 1.1: comparison with reference solution

Mpfa-dfm

Reference solution [4]
Mpfa - dfm

Test case 1.2: heterogeneous fracture permeability
Mpfa – dfm

Test case 2: impermeable fracture tips
Mpfa - dfm

Test case 3: flux and pressure continuity on fracture tips

\[ \alpha_i = n_{tip}^T K_{mi} \frac{x_i - x_{tip}}{||x_i - x_{tip}||^2} \]

\[ w_i = \frac{\alpha_i}{\sum_i \alpha_i} \]

\[ q_i = w_i f_{tip} \]
Mpfa - dfm

Test case 3: flux and pressure continuity on fracture tips
Mpfa - dfm

Approach 2: dirichlet coupling on fracture boundaries

\[-\xi v_A n_A + \alpha_f \bar{u}_A = -(1 - \xi) v_B n_B + \alpha_f \bar{U}_f\]

\[
(\xi = 1) \quad \rightarrow \quad \alpha_f (\bar{u}_A - \bar{U}_f) = v_A n_A
\]

\[
\alpha_f (\bar{u}_A - \bar{U}_f) = \sum_j T_{iA} u_j
\]
**Mpfa - dfm**

*Approach 2: dirichlet coupling on fracture boundaries*

\[
\alpha_f(\bar{u}_A - \bar{U}_f) = \sum_{j \notin \{j_A, j_B, j_C, j_D\}} T_{i Aj} u_j + \sum_{j \in \{j_A, j_B, j_C, j_D\}} T_{i Aj} \bar{u}_j
\]

\[
\alpha_f(\bar{u}_A - \bar{U}_f) = \beta_A + \sum_{j \in \{j_A, j_B, j_C, j_D\}} T_{i Aj} \bar{u}_j
\]

\[
(T_{i Aj_A} - \alpha_f) \bar{u}_{jA} + T_{i Aj_B} \bar{u}_{jB} + T_{i Aj_C} \bar{u}_{jC} + T_{i Aj_D} \bar{u}_{jD} = \alpha_f \bar{U}_f - \beta_A
\]
Mpfa - dfm

Approach 2: dirichlet coupling on fracture boundaries

Local system of equations for the face potentials:

\[
(T_{iA} - \alpha_f) \bar{u}_j + T_{iB} \bar{u}_j + T_{iC} \bar{u}_j + T_{iD} \bar{u}_j = \alpha_f \bar{U}_f - \beta_A
\]

\[
T_{iB} \bar{u}_j + (T_{iB} - \alpha_f) \bar{u}_j + T_{iC} \bar{u}_j + T_{iD} \bar{u}_j = \alpha_f \bar{U}_f - \beta_B
\]

\[
T_{iC} \bar{u}_j + T_{iC} \bar{u}_j + (T_{iC} - \alpha_f) \bar{u}_j + T_{iD} \bar{u}_j = \alpha_f \bar{U}_f - \beta_C
\]

\[
T_{iD} \bar{u}_j + T_{iD} \bar{u}_j + T_{iD} \bar{u}_j + (T_{iD} - \alpha_f) \bar{u}_j = \alpha_f \bar{U}_f - \beta_D
\]
Mpfa – dfm

Approach 3: internal flux boundaries on fracture facets

\[-\xi v_{n,\gamma_1} n_{\gamma_1} + \alpha_f \bar{u}_{\gamma_1} = -(1 - \xi) v_{n,\gamma_2} n_{\gamma_2} + \alpha_f \bar{U}_f,\]

\[-\xi v_{n,\gamma_2} n_{\gamma_2} + \alpha_f \bar{u}_{\gamma_2} = -(1 - \xi) v_{n,\gamma_1} n_{\gamma_1} + \alpha_f \bar{U}_f,\]

Local system for $\xi = 1$:

\[
f_i = \omega_{ij+1}(\bar{u}_{j+1 \rightarrow i} - u_j) + \omega_{ij+2}(\bar{u}_{j+2 \rightarrow i} - u_j) = \omega_{ij-1}(\bar{u}_{j-1 \rightarrow i} - u_j) + \omega_{ij-2}(\bar{u}_{j-2 \rightarrow i} - u_j),
\]

\[
f_{N+} = \omega_{iN+j+1}(\bar{u}_{N+} - u_j) + \omega_{iN+j+2}(\bar{u}_{j+2 \rightarrow i} - u_j) = \alpha_f(\bar{u}_{N+} - \bar{U}_f),
\]

\[
f_{N-} = \omega_{iN-j+1}(\bar{u}_{N-} - u_j) + \omega_{iN-j+2}(\bar{u}_{j+2 \rightarrow i} - u_j) = \alpha_f(\bar{u}_{N-} - \bar{U}_f).
\]

\[
\rightarrow f = (CA^{-1} B - D) u + CA^{-1} \bar{f}
\]
Mpfa - dfm
*Test case 1.3: low fracture permeability*
Mpfa – dfm

Test case 1.3: heterogeneous low fracture permeability
Mpfa - dfm

Test case 4: crooked barrier
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Motivation

Multi-point flux approximation methods

Mpfa - discrete fracture model

Summary and Outlook
Summary and Outlook

Overview over the introduced method:

- FV Mpfa model in matrix domain
- Extended local systems in matrix \(\rightarrow\) fractures are interior boundaries
- Fracture-matrix interaction included through coupling conditions...
  - as source terms in fracture domain
  - in the local systems in matrix domain
- The method is...
  - locally mass conservative
  - consistent for heterogeneous porous media
Summary and Outlook

Outlook:

- Implementation of a monolithic scheme
  → parameter & convergence study
  → study on influence of the parameter $\xi$
- Extension to multiphasic flow and transport
- Implementation of MPSA methods for linear elasticity & Biot
  → inclusion of fracture propagation models
    → re-meshing algorithm development
- Extension to 3D ...
THANKS FOR YOUR ATTENTION

ANY QUESTIONS?
References

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